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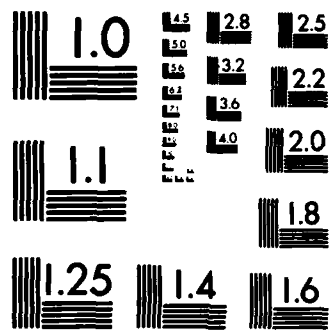
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AN ALGEBRAIC AND GRAPHICAL STUDY

BY

W. FOODY AND A. HEDAYAT

TECHNICAL REPORT IN STATISTICS

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ABSTRACT

The structure and the size of the supports of balanced incomplete block (BIB) designs are explored. The concept of *fundamental* BIB designs is introduced and its usefulness in the study of the support of BIB designs is demonstrated. It is shown that the support size can be recuced via a technique called *trade on a design*. A new graphical method of studying the supports of BIB designs with blocks of size three is introduced. Several useful results are obtained via this graphical method. In particular, it is shown that no BIB design with seven varieties in blocks of size three can be built based on sixteen distinct blocks. Contributions made here have immediate applications in controlled experimental designs and survey samplings.

0. INTRODUCTION.

The standard statistical optimality of balanced incomplete block (BIB) designs has nothing to do whether or not the design has repeated blocks. However, it is known that such designs have interesting additional applications in design of experiments and controlled survey sampling. Therefore, it is useful to study the existence and nonexistence of BIB designs with repeated blocks and catalog them for practical applications. The set of distinct blocks, referred to as the support, of BIB designs plays a crucial role in the study of BIB designs. This paper is mainly devoted to explore the structure of the supports for this family of designs.

Formal definitions and notations are introduced in section 1. In section 2 we shall briefly present the algebraic formulation of all BIB designs by Foody and Hedayat (1977). This allows us to introduce, for the first time, the concept of fundamental BIB designs and demonstrate its usefulness in our study. The general concept of trades of Hedayat and Li (1979) will be restricted to the notion of trades on a design. We have utilized this latter idea for reducing the support size of a design.

The structure of supports and their possible sizes are studied in detail in section 3. Some implications of our results on the entire design are also pointed out. A graphical description of the support for blocks of size 3 is introduced and studied in section 3.3. We have demonstrated the usefulness of our graphical description by applying the result to the case of $v = 7$, $k = 3$

in section 4. These techniques allowed us to conclude that it is impossible to build a BIB design based on 7 varieties in blocks of size 3 if we are limited to have 16 distinct blocks only.

1. DEFINITIONS AND NOTATION.

Let $v\Sigma k$ be the set of $\binom{v}{k}$ distinct subsets of size k based on the set $V = \{1,2,\ldots,v\}$. We will refer to the elements of V as varieties. For convenience the number $\binom{v}{k}$ will be denoted by vCk. A balanced incomplete block (BIB) design with parameters v,b,r,k,λ,b^* is a collection of b elements of $v\Sigma k$, referred to as blocks, with properties:

i) each variety occurs in exactly r blocks,

ii) each pair of distinct varieties appears together in exactly λ blocks,

iii) there are exactly b^* distinct blocks among all b blocks of the design.

If $b^* < b$ then we say the design is a BIB design with repeated blocks. The support of a BIB design, D, is the collection of distinct blocks in D, denoted by D^*. We will denote the cardinality of D^* by b^* and shall refer to b^* as the support size of D.

We will denote a BIB(v,b,r,k,λ) with support size b^* by BIB$(v,b,r,k,\lambda|b^*)$. Any incomplete block design may be specified by the number of times that each element of $v\Sigma k$ is repeated in that design. We write f_i for the frequency of the ith element of $v\Sigma k$ in the design. Thus, we identify an incomplete block

design, D, with $v\Sigma k$ and the frequency vector $F = (f_1, f_2, \ldots, f_{vCk})'$. It is clear that $b = f_1 + f_2 + \ldots + f_{vCk}$ and that b^* is the number of non-zero entries in the vector F. The BIB design, D, is said to be a uniform BIB design if the non-zero components of F are all identical. A BIB design with $b = b^* = vCk$ is denoted by $DT(v,k)$ and referred to as the <u>trivial BIB design</u> based on v and k. A BIB design with $b < vCk$ is said to be a <u>reduced</u> design.

2. CONSTRUCTION OF BIB DESIGNS WITH REPEATED BLOCKS.

In this section we list techniques for constructing BIB designs with repeated blocks from already known BIB designs. The requirement that we begin with a known design is not unduly restrictive, since for any v and k the trivial design is available. From the trivial design we will be able to construct many other designs.

2.1. P-matrix Representation of BIB Designs.

To introduce the new concept of <u>fundamental</u> BIB designs we need some algebraic results and ideas of Foody and Hedayat (1977) which will be introduced first. Given v and k, begin by labelling the elements of $v\Sigma 2$ from 1 to $vC2$ and those of $v\Sigma k$ from 1 to vCk. Let $p_{ij} = 1$ if the ith element of $v\Sigma 2$ is contained in the jth element of $v\Sigma k$, and let $p_{ij} = 0$ otherwise-

wise. Let P be the matrix (p_{ij}). Thus P is an incidence matrix relating pairs and k-sets in the trivial design for v and k.

Since any incomplete block design can be identified with its frequency vector, F, we will often refer to "the BIB design F", meaning the design determined by F. It is easy to verify:

__Lemma 2.1__. __The frequency vector__ F __determines a__ BIB __design if and only if__

$$PF = \lambda \underline{1} \tag{2.1}$$

__where__ λ __is a positive integer__.

If $F = (f_1,\ldots,f_n)'$ and $G = (g_1,\ldots,g_n)'$ are vectors, we will write $F > G$ if $f_i \geq g_i$ for all i and $f_j > g_j$ for some j. Therefore, the problem of constructing all BIB designs based on v, k, and λ is precisely the problem of finding all non-negative integer solutions, F, to the equation $PF = \lambda\underline{1}$. In the language of mathematical programming, we want to solve the system

$$\begin{aligned} PF &= \lambda\underline{1} \\ F &\geq \underline{0} \end{aligned} \tag{2.2}$$

for integer values of F. If we are not interested in a particular value of λ, then this integer programming problem may be replaced by the linear programming problem of finding rational solutions to (2.2). Multiplying both λ and F by a common multiple of the denominators of the entries of F will give a new frequency vector of integers and a new λ which will fulfill condition (2.1).

Lemma 2.1 will now be used to give a *geometric* characterization of the set, $\mathcal{F}$ of all BIB designs for a given v and k.

Proposition 2.2. *If* $c_1,\ldots,c_n$ *are non-negative integers, not all equal to* 0, *and if* $F_1,\ldots,F_n$ *are in* $\mathcal{F}$, *then*

$$c_1F_1 +\ldots+ c_nF_n \text{ is in } \mathcal{F}.$$

A set with the property which Proposition 2.2 ascribes to $\mathcal{F}$ is called a positive integer cone. Lemma 2.1 also gives immediately the following fact about $\mathcal{F}$.

Proposition 2.3. (i) *If* $F \in \mathcal{F}$ *and* g *is a common divisor of the entries of* F *then* $g^{-1}F \in \mathcal{F}$. (ii) *If* F_1 *and* F_2 *are in* $\mathcal{F}$ *with* $F_1 > F_2$ *then* $F_1 - F_2 \in \mathcal{F}$.

Note that it follows from (i) that if there is no BIB design with $b < vCk$, then there is no uniform BIB design with $b^* < vCk$. It is clear that for any fixed λ there are only finitely many solutions to (2.1), and that if λ is free to vary, there are infinitely many. But many of these solutions are in fact positive combinations of other ones. It is shown that there are only finitely many designs that are not such combinations. To be more precise, we make the following definition. A BIB design F is a *fundamental design* if there does not exist any BIB design F_1 such that $F > F_1$.

Corresponding to the concept of a fundamental design is that of an irreducible solution in non-negative integers to a system of homogeneous linear equations. Consider the set of non-negative integer solutions to the set of homogeneous linear equations

$$AX = \underline{0} \tag{2.3}$$

where A is an $m \times n$ matrix of integers. Such a solution X_1 is called <u>irreducible</u> if for no other such solution, X_2, $X_1 > X_2$. It is known that there are only finitely many irreducible solutions to (2.3). For a proof of this fact see, for example, Grace and Young [(1903)]. Notice that, for a given v and k, the vector F determines λ and that if $F_1 < F_2$ then $\lambda_1 < \lambda_2$. Thus, each fundamental BIB design corresponds to an irreducible solution to the system

$$\left[P \,\vdots\, -\underline{1}\right]\begin{bmatrix} F \\ \cdots \\ \lambda \end{bmatrix} = \underline{0}. \tag{2.4}$$

Therefore we have

Proposition 2.4. For any given v and k there are only finitely many fundamental BIB designs.

It is worth noting that every fundamental designs is a basic feasible solution to the integer program (2.2) for some value of λ, but that the converse does not hold. For example, if F_1 is a fundamental design with parameter v,b,r,k,λ then $2F_1$ will still be a basic feasible solution to the program.

$$PF = (2\lambda)\underline{1}$$

$$F \geq \underline{0} .$$

The fundamental designs are fundamental in the sense that they generate all BIB designs for a given v and k.

Proposition 2.5. For a given v and k let F be the frequency vector of a BIB design and let $F_1,\ldots,F_n$ be the frequency vectors of the fundamental designs. Then there exist non-negative integers $a_1,\ldots,a_n$ such that

$$F = a_1F_1 + \ldots + a_nF_n \tag{2.5}$$

Proof: If b is minimal for v and k, then F is clearly fundamental and 2.5 is satisfied. Proceeding by induction on b, suppose F is the frequency of an arbitrary design D. If D is fundamental, then 2.5 is satisfied. If D is not fundamental, then there exists a design with frequency vector F_1

such that $F_1 < F$. By Proposition 2.3, $F - F_1$ is also the frequency vector of a BIB design and both $\underline{1}'F_1$ and $\underline{1}'(F-F_1)$ are less than b. Thus, by the inductive hypothesis, both F_1 and $F - F_1$ have representations as in 2.5. But so then does F, since $F = F_1 + (F-F_1)$.

2.2. Construction of BIB Designs by Trades

This section discusses the construction of one BIB design from another by trading blocks. Suppose that the BIB design D contains a set of (not necessarily distinct) blocks, S. Suppose also that there exists another set of blocks, S', based on the same v and k such that S and S' contain the same pairs of varieties the same number of times. If we remove S from D and replace it by S', then the new design will still be a BIB with the same parameters v,b,r,k,λ, but with possibly a different value of b^*. Following Hedayat and Li (1979), we define a trade in terms of the P-matrix discussed above: A non-zero vector of integers, T, is called a trade if $PT = 0$.

Note that this definition makes no reference to any particular design, but depends only on the parameters that define P, namely v and k. The following lemma is due to Hedayat and Li (1979).

Lemma 2.6. *Let* $F = (f_1,\ldots,f_{vCk})'$ *be the frequency vector of a* BIB(i.e., $PF = \lambda\underline{1}$) *and let* $T = (t_1,\ldots,t_{vCk})'$ *be a*

trade (i.e., $PT = \underset{\sim}{0}$).

(i) *For all positive integers* m *and* n, $mF + nT$ *is a* BIB *design if and only if* $mF + nT > 0$.

(ii) *The condition that* $f_i > 0$ *whenever* $t_i < 0$ *is necessary and sufficient for there to exist positive integers* m *and* n *such that* $mF + nT$ *is a* BIB *design*.

Notice that for any trade T, $t_1 + \ldots + t_{vCk} = 0$; that is

$$0 = \underset{\sim}{1}'\underset{\sim}{0} = \underset{\sim}{1}'(PT) = (\underset{\sim}{1}'P)T = (kC2)\underset{\sim}{1}'T.$$

So T has both positive and negative entries. In Hedayat and Li (1979) the sum of the positive entries is called the *volume* of the trade.

Trades for which the blocks added and the blocks subtracted are both already present in the support of a design play an important role in the sequel. We will say that a trade $T = (t_1, \ldots, t_{vCk})'$ is a *trade on the design* D if $t_i = 0$ for all blocks not in D^*.

Trades on the design D may also be characterized as follows: remove from the P-matrix all columns corresponding to blocks absent from D^*, and call the resulting matrix P^*. Then each vector T^* of integers satisfying $P^*T^* = \underset{\sim}{0}$ corresponding to a trade, T, on the design D. To reconstruct T from T^*, let $t_i = 0$ if the i-th block is absent from D^*, and $t_i = t_j^*$ if the i-th block corresponds to the j-th column of P^*. So sets of trades on a given design, like sets of trades in general, correspond to the integer valued vectors

in the null space of a matrix. By utilizing some results of Foody and Hedayat (1977) we have:

Proposition 2.7. *Let F be the frequency vector of the design D. If T is a trade on the design D, then there exist positive integers m and n such that $mF + nT$ is the frequency vector of a BIB design whose support is properly contained in D^*.*

Proof: Let $F = (f_1,\ldots,f_{vCk})'$ and $T = (t_1,\ldots,t_{vCk})'$. Also select j so that

$$t_j/f_j = \min\{t_i/f_i \mid f_i \neq 0\}.$$

Then $t_j < 0$, so there exist positive integers m and n such that

$$mf_j + nt_j = 0.$$

But

$$mf_i + nt_i \geq f_i f_j^{-1}(mf_j + nt_j) = 0 \qquad i = 1,\ldots,vCk.$$

Some value of t_i is positive, so not all $mf_i + nt_i$ are equal to 0. Thus $mF + nT$ defines a BIB design by Lemma 2.6, and its support is a proper subset of D^*.

Proposition 2.8. *If D and D_1 are BIB designs such that D_1^* is properly contained in D^*, then there exists a trade on D.*

<u>Proof:</u> Let F and G be the frequency vectors of D and D_1, respectively. Then $PF = \lambda \underline{1}$ and $PG = \lambda_1 \underline{1}$, and there exist positive integers m and n such that $m\lambda - n\lambda_1 = 0$. Then $P(mF-nG) = \underline{0}$, and $mF - nG \neq \underline{0}$, since for some i, $f_i > 0$ but $g_i = 0$. Thus $mF - nG$ is a trade. And since $D_1^* \subseteq D^*$, it is a trade on D.

Starting with the trivial design for v and k, a trade can be constructed by finding a non-zero rational solution to the equation $PT = \underline{0}$. The solution vector is then multiplied by the least common multiple of its denominators to give a vector of integers, that is, a trade. This trade is applied to a multiple of the trivial design to produce a new design, D_1, as in Proposition 2.7, with smaller support. Now remove from P the columns corresponding to blocks absent from D_1^* to produce P_1. Find a solution to $P_1T = \underline{0}$ and continue as above. This process will ultimately produce a design whose support cannot be reduced. Foody and Hedayat (1977) utilized a result similar to Proposition 2.7 and presented some techniques for producing BIB designs whose support is contained within a given design.

3. THE SUPPORT OF A BIB DESIGN.

This section is concerned with the supports of BIB designs. First we examine the characteristics of designs whose supports are minimal. We then provide some lower bounds for the size of a support, and consider some conditions under which sets of blocks may form the support of a BIB design. The case where the block size, k, is equal to three will be discussed in greater detail.

3.1. Minimal Supports

For a given v and k we can partially order by set inclusion all of the supports of BIB designs based on that v and k. Let us refer to the minimal elements under this ordering as minimal supports. The discussion after Proposition 2.8 provides a technique for generating designs with minimal supports.

The following proposition shows that all BIB designs with the same minimal support are, in a sense, the same.

Proposition 3.1. Let D be a BIB design based on v and k. Then D has minimal support if and only if any other BIB design with the same support is a rational multiple of D.

Proof: Suppose D_1 is also a BIB design and let the frequency vectors of D and D_1 be F and G respectively.

Suppose that D^* is a minimal support and that $D_1^* = D^*$ but D_1 is not a multiple of D. Now $PF = \lambda \underline{1}$ and $PG = \lambda_1 \underline{1}$ for positive integers λ and λ_1, and there exist positive integers q and s such that $q\lambda - s\lambda_1 = 0$. Let $T = qF - sG$. Since D_1 is not a multiple of D, it follows that $T \neq \underline{0}$. Thus T is a trade on the design D, and by Proposition 2.7, D^* cannot be minimal.

To show the converse, suppose that every design with support D^* is a rational multiple of D, but that D^* is not a minimal support. Then there exists a design D_1 such that D_1^* is properly contained in D^*. If F and G are defined as above, clearly G is not a rational multiple of F. Thus $nF - G$ is never a rational multiple of F. But for a large enough integer n, the support of the design defined by $nF - G$ is D^*, since $nF - G > F$ and $D_1^* \subseteq D^*$. This is a contradiction.

Corollary 3.2. For a given v *and* k *let* D^* *be a minimal support and let* $\mathcal{S}$ *be the set of all BIB designs supported by* D^*. *Then there exists a unique design* $D \in \mathcal{S}$ *such that all other designs in* $\mathcal{S}$ *are integer multiples of* D. *Further,* D *is a fundamental design.*

Proof: Choose a design D in $\mathcal{S}$ with the smallest value of λ out of all designs in $\mathcal{S}$, and let F be its frequency vector. Note that the greatest common divisor of the entries

of F is one, by Proposition 2.3. Thus, if aF is a design for some rational number a, it follows that a must be an integer. So, by the last proposition, all designs supported by D^* are integer multiples of D. Also, if some integer multiple of D is to have the same value of λ as D does, then this other design must be equal to D, which demonstrates uniqueness. Finally, if G is the frequency vector of a design such that $G < F$, then the support of this design must be equal to D^*, by minimality of D^*. But $G < F$ implies that $PG < \lambda\underline{1}$, contradicting the construction of D. Thus D is a fundamental design.

An interesting problem, for which we do not know the answer, is how to find the values of v and k for which all fundamental designs have minimal supports.

For any v and k we can give an upper bound on the number of blocks in a minimal support. In fact, this bound depends only on v.

<u>Proposition 3.3</u>. <u>For any given</u> BIB <u>design with minimal support</u>, $b^* \leq vC2$.

<u>Proof</u>: If D has minimal support, then by Proposition 2.7, there does not exist a trade on D, i.e., if $F = (f_1,\ldots,f_{vCk})'$ is the frequency vector of D, there is no vector $T = (t_1,\ldots,t_{vCk})'$ such that $t_i = 0$ whenever $f_i = 0$ and

$PT = \underline{0}$. Equivalently, forming P^* by removing the columns of P corresponding to blocks for which $f_i = 0$, there is no vector T^* such that

$$P^*T^* = \underline{0}.$$

That is, P^* has full column rank. Thus the number of columns of P^*, namely b^*, must be less than or equal to the number of rows, namely $vC2$.

3.2. Lower Bounds on the Support Size of a BIB Design.

Lower bounds for b, the number of blocks in a BIB design are well known. One such result is Fisher's inequality: $b \geq v$. In this section we give some lower bounds on b^*, the number of *distinct* blocks in the design. Some of these bounds depend upon an inequality due to Mann (1969).

Lemma 3.4 (Mann). If $F = (f_1,\ldots,f_{vCk})'$ *is the frequency vector of a* BIB(v,b,r,k,λ), *then* $f_i \leq b/v$, $i = 1,\ldots,vCk$.

Utilizing Mann's inequality we obtain the following useful Corollary.

Corollary 3.5. *If* $F = (f_1,\ldots,f_{vCk})'$ *is the frequency vector of* D, *a* BIB(v,b,r,k,λ), *and if* $f_i = b/v$ *for some* i, *then every block in* D^* *intersects the* i-th *block in the same number of elements*.

For example, if $v = 7$ and $k = 3$ then the basic necessary conditions on the parameters show that $\lambda = b/v$. Thus, any block with frequency λ intersects every other block in the support in exactly one variety.

From Mann's inequality we obtain the following corelatives of Fisher's inequality.

Proposition 3.6. $b^* \geq v$.

Proof: By Mann, $b/v \geq f_i$. Summing on both sides over all non-zero values of f_i, we get

$$b^*(b/v) \geq b,$$

giving the result desired.

Proposition 3.7. *If* $b^* = v$ *then the design is uniform.*

Proof: First $b/v = \Sigma f_i/v$, summing over the non-zero entries in F. If $b^* = v$, this implies that $b/v = \Sigma f_i/b^*$; that is, that b/v is equal to the average of the non-zero entries in F. But by Mann, b/v is greater than or equal to *each* of

these entries. Thus, they all must be equal to b/v, and so to each other.

These last two propositions have also been proved by van Lint and Ryser (1972), using a different technique. In the same article, van Lint and Ryser also proved (essentially) the following proposition.

Proposition 3.7. *In a BIB design,* $b^* \neq v + 1$. *Thus, in a non-uniform BIB design,* $b^* \geq v + 2$.

Obviously in a BIB design the frequency of any block cannot exceed λ.

Proposition 3.8. *Suppose* $(f_1, \ldots, f_{vCk})'$ *is the frequency vector of a* BIB(v,b,r,k,λ). *Then*

(i) $f_1 \leq \lambda$,

(ii) $b^* \geq b/\lambda$,

(iii) If $b^* = b/\lambda$ then the design is uniform.

It is worth noting that this last bound for b^* is independent of b, since

$$b/\lambda = v(v-1) / (k(k-1)).$$

A slightly better version of this last bound for b^* can be produced. Let $\lceil x \rceil$ be the smallest integer greater than or equal to x. Foody and Hedayat (1977) proved:

Proposition 3.9. $b^* \geq \lceil (v/k) \lceil (v-1)/(k-1) \rceil \rceil$.

Before determining what happens when equality obtains in Proposition 3.9, some additional notation and terminology will be introduced. If $F = (f_1,\ldots,f_{vCk})'$ is the frequency vector of a design D for a given v and k and if B is the i-th element in the ordering of the blocks, then $f(B) = f_i$. If $X \subset \{1,\ldots,v\}$ then $s(X)$ is the number of distinct blocks in D^* containing X.

A set S of distinct blocks is a *covering of the pairs* if every pair of varieties is contained in S. S is a *minimal covering* (of the pairs) if no proper subset of S is also a covering of the pairs.

What has been shown in Proposition 3.9 is that every minimal covering must contain at least $\lceil (v/k) \lceil (v-1)/(k-1) \rceil \rceil$ distinct blocks. Before completing the discussion of minimal coverings, the following simple lemma of Foody and Hedayat (1977), to be used many times in the sequel, is presented.

Lemma 3.10. *If* X *and* Y *are pairs of varieties, both contained in the same block* B *of a* BIB *design, and if* $s(X) = 1$, *then* $s(Y) = 1$.

Proposition 3.11. Suppose S is a minimal covering of pairs and D is a BIB design such that $D^* = S$. Then D is a uniform design.

Proof: First, we show that the design produced by taking one copy of each block in S is a BIB design. For, if it is supposed otherwise, then not every pair of varieties occurs exactly once in S, since this would guarantee a design with $\lambda = 1$. But every pair occurs in at least one block of S, since S is a covering of the pairs. So, there exists a pair Y such that $s(Y) > 1$. Now if B is a block containing Y, then for every other pair X in B, $s(X) > 1$ by Lemma 3.10. Then $S - \{B\}$ is a covering of the pairs, contradicting the minimality of S. Thus S is itself a BIB design.

S, considered as a BIB design, certainly is a minimal support, and every non-zero frequency is equal to one. Thus by Proposition 3.1, D is a multiple of S, proving the result.

Corollary 3.12. If D is a BIB design such that $b^* = \lceil (v/k)\lceil (v-1)/(k-1)\rceil\rceil$, then D is a uniform design.

We now have two lower bounds for b^*, namely v and $\lceil (v/k)\lceil (v-1)/(k-1)\rceil\rceil$. These bounds can be attained. For example, in the BIB(7,7,3,3,1|7), these bounds are equal to each other and to b^*. In general, if $v \geq k^2 - k + 1$ then $v \leq \lceil (v/k)\lceil (v-1)/(k-1)\rceil\rceil$. If, one the other hand, v is small

compared to $k^2 - k + 1$, then the bound given in Proposition 3.6 is sharper. It may be, however, that for a given v and k neither of these bounds is achieved.

If $v < k^2 - k + 1$ there is another bound for b^* which is sometimes sharper than that given by Proposition 3.6.

Proposition 3.13. *Suppose* D *is a* BIB$(v,b,r,k,\lambda|b^*)$ *and that* $v < k^2 - k + 1$. *Then*

(i) $b^* \geq \frac{v}{k}\left\{\frac{2(v-1)}{k-1}\right\}$, *and further*

(ii) *if* $b^* = \frac{v}{k}\left(\frac{2(v-1)}{k-1}\right)$ *then* D *is uniform*.

Proof: It is easy to see that $v < k^2 - k + 1$ is equivalent to $b/v < \lambda$. Thus, by Mann's inequality, the frequency of every block is strictly less than λ, so every pair of varieties must occur in at least two blocks of D^*.

For any given variety, x, each of the $v - 1$ pairs xy must have $s(xy) \geq 2$. Therefore $s(x) \geq 2(v-1)/(k-1)$. But, as we argued in the proof of Proposition 3.11, on the average each variety occurs in b^*k/v blocks of D^*, and the fact that the average is at least as great as the minimum gives result (i).

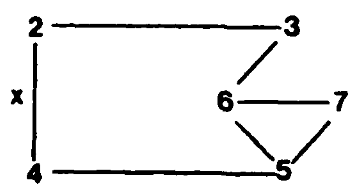
By this analysis, if $b^* = \frac{v2(v-1)}{k(k-1)}$, then the average variety, and hence every variety, occurs in the minimal number of blocks of D^*, namely $2(v-1)/(k-1)$. Thus $s(xy) = 2$ for every pair of varieties, xy, and D^* is itself a uniform BIB design.

But certainly D^* is a minimal support, since it achieves the lower bound in (i), and so by Proposition 3.1, D is uniform.

Consider the case when $v = 6$ and $k = 3$. Part (i) of this last proposition says that $b^* \geq 10$ and part (ii) says that if $b^* = 10$ then the design is uniform. It is well known that there is a uniform BIB design with these parameters. Notice that Propositions 3.6 and 3.9 each give a bound of six for b^* in this case. For further results on $v = 6$ and $k = 3$ see Hedayat and Khosrovshahi (1981).

3.3. Graphical Description when $k = 3$.

If we restrict our attention to block designs in which each block contains three varieties, we can describe the supports of these designs by means of graphs. In particular, if a is a variety in a block design D for which $k = 3$ define the <u>figure around a in D</u> to be the adjacency graph whose vertices are the other $v - 1$ varieties. Two vertices, b and c, are adjacent if $\{a\ b\ c\} \in D^*$. For example, if the figure around 1 is

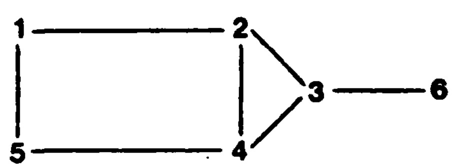


then the blocks of the design containing variety 1 are 124, 123, 145, 136, 167, 156, 157. Since each line of the graph represents a block, we will sometimes indicate on the graph the frequency of the block. In the example above, $f(124) = x$.

If D is a BIB design, and if b is a vertex of the figure around a in D, then the degree of b is just s(ab). Also the sum of the frequencies of all lines incident with a vertex, called the <u>index</u> of the vertex, must be λ for each vertex.

Given a set of distinct blocks, S, we can draw the figures around each variety without assigning frequencies to the edges. Certainly a necessary condition for S to support a BIB design is that each figure be <u>balanceable</u>, that is, that there exist an assignment of positive integers to the edges of the graph in such a way that the sum of the integers on all lines incident with a vertex be the same for all vertices.

For example, the following figure cannot be balanced:



This is clear, since whatever positive integer is assigned to the edge 3-6 will be the total for vertex 6, and thus for all vertices. But the total for vertex 3 must exceed this, since edge 3-4 must be assigned a positive integer frequency. Thus, if the above graph were the figure around variety 7 in some set of blocks, that set could not possibly be the support of a BIB design.

We will set forth in this sub-section a few propositions giving conditions which would guarantee that a graph not be balanceable, and we will apply these propositions in the next section to show the non-existence of certain designs.

There is a literature on the subject of balancing graphs (see, for example, Stewart (1966), Kotzig and Rosa (1973), and Stanley (1976)). In this literature an assignment of frequencies to a graph in a balanced way is called a "magic labeling" of the graph, due to the relation of these graphs to magic squares. Tutte (1952) has given a necessary and sufficient condition for a graph to be unbalanceable, but his concept of balancing allows frequencies of zero, which would not be helpful in our context. Stanley (1973) rewrites a theorem of Stiemke (1915) on diaphantine equations into (essentially) the following condition for balancing a graph:

A finite graph cannot be balanced if and only if there exists a labeling $K:V \to Z$ of the vertices of G by integers such that $\Sigma_{v \in V} K(v) \leq 0$ and for each edge, e, $\Sigma_{v:v \in e} K(v) \geq 0$, with at least one of these sums not equal to 0.

This result is stronger but less easy to apply than the propositions which we prove below.

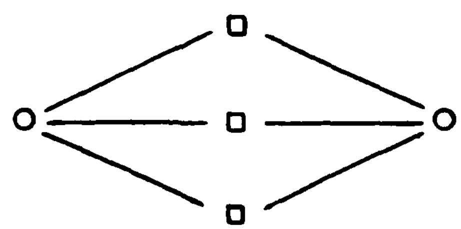
First, let us recall some graph theoretic terminology. A graph is said to be <u>bi-colorable</u> if the vertices can be divided into two disjoint sets, say reds and green, such that no two adjacent vertices are of the same color. A connected subset X of a graph forms a <u>component</u> if no vertex in X is adjacent to any

vertex outside of X. A <u>subgraph</u> of a graph is a subset of the set of vertices, along with a subset of the lines connecting them. We call a sequence of distinct vertices in which each vertex is adjacent to the one preceeding it and the first is adjacent to the last a <u>cycle</u>. The <u>length</u> of the cycle $\{a_1,\ldots,a_n\}$ is n, and the <u>distance</u> between a_i and a_j, with $j \geq i$ is $\min\{j-i,\ n-j+1\}$.

<u>Proposition 3.14</u>. <u>Suppose</u> G <u>is a balanceable graph and some component of</u> G <u>is bi-colorable</u>. <u>Then the number of vertices of each color is the same</u>.

<u>Proof</u>: We can assign positive frequencies to each edge of G so that the index of each vertex is the same, say λ. Restrict our attention to the bi-colorable component and consider only the vertices of one of the colors. Now every edge in the component is incident with one and only one of the vertices of that color. Thus the sum of all of the indices of vertices of that color is equal to the sum of the frequencies of all of the dges in the component. But the same is certainly true of the other color. But for each color, the total of the indices is simply the number of vertices times λ. Thus, the number of vertices is the same for both colors.

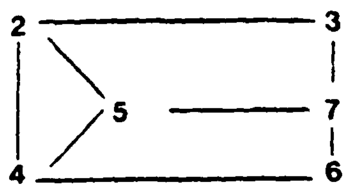
As an example, the following graph cannot be balanced:



Proposition 3.15. Suppose that G is a balanceable graph and that some subgraph H of G is bi-colorable, with the same number of vertices of each color. If for one color the degree of every vertex is the same in H as in G, then the same is true for the other color.

Proof: Suppose that in H there are n vertices of each color, and that the index of the reds is the same in H as in G, say λ. Then as in the proof of the last proposition, the total frequency of all of the edges in H must be $n\lambda$, and the total index in H of the greens must be $n\lambda$. But no vertex may have an index in H higher than that in G, namely λ. Thus every green vertex has index λ in H. But if there were any edge of $G - H$ incident with a green vertex, this would raise its frequency in G to more than λ, which is a contradiction of the fact that G is balanceable.

For example let the figure around variety 1 be



Then these blocks cannot support a BIB design. For, if we remove line 24, then the remaining subgraph is bi-colorable with, say $\{2,4,7\}$ as the greens and $\{3,5,6\}$ as the reds. Notice that the degree of each red vertex is the same in the

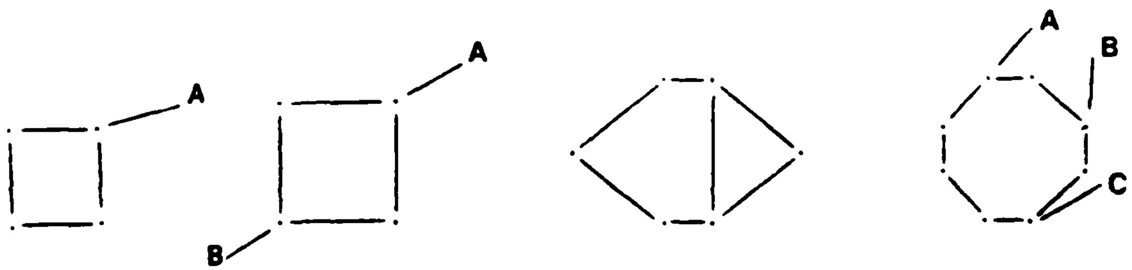
subgraph as in the original figure, but that this is not true for the greens. Thus, the proposition is violated.

The following useful corollary is a special case of the last proposition.

Corollary 3.16. Suppose G *is a balanceable graph and* H *is a cycle in* G. *Suppose* H *has even length and that* T *is a subset of the vertices of* H *in which every vertex is an even distance from every other. If every vertex of* $H - T$ *has degree* 2, *then every vertex of* T *has degree* 2.

Proof: Since H is a cycle of even length, it is bi-colorable with the same number of vertices of each color. Also T is entirely of one color, so all of the vertices of the other color are in $H - T$, and thus have the same degree in H as in G. The proposition can now be applied.

This corollary shows that the following cannot be figures around a variety in a BIB design. Here A, B, and C represent subgraphs.



The following fact is just a restatement of the elementary Lemma 3.10. It is also a special case of Proposition 3.15.

Corollary 3.17. If G is a balanceable graph then every vertex of degree one is adjacent only to another of degree one.

To conclude this sub-section, we state a fact about the number of different frequencies possible in very simple graphs.

Proposition 3.18. If G is a graph of index λ and if a component, H, of G is a cycle, then there is a positive integer x such that every edge has frequency x or $\lambda - x$. Moreover, if H has an odd number of vertices, then $x = \lambda/2$.

Proof: Suppose H has n vertices. We can then sequentially label the edges of H from 1 to n following a path around the cycle. If x is the frequency of the edge number 1, then edge 2 must have frequency $\lambda - x$. Arguing inductively, it is easy to see that odd numbered edges have frequency x and all even numbered edges have frequency $\lambda - x$. If n is odd, then edge n and edge 1 are both incident with vertex n, and both have frequency x. Thus, in this case, $x = \lambda/2$.

4. BIB$(7,b,r,3,\lambda|b^*)$.

The case of $v = 7$, $k = 3$ has been investigated by Hedayat and Li (1979, 1980) in regard to the possible combinations of b and b^*. Key results in their 1979 paper can be summarized this way. There exists a BIB$(7,b,r,3,\lambda|b^*)$ if and only if (i) $b \equiv 0 (\mathrm{mod}\ 7)$; (ii) $7 \leq b^* \leq \min(b,35)$; (iii) $b^* \neq 8,9,10,12$, or 16; (iv) $(b,b^*) \neq (28,24)$, $(28,27)$, $(35,30)$, $(35,32)$, $(35,33)$, $(35,34)$, or $(42,34)$. These authors did not give a proof for the nonexistence of a BIB$(7,b,r,3,\lambda|16)$, instead they made reference to others. The story of $b^* = 16$ is this. Seiden (1977) proved that based on 21 blocks it is impossible to build a BIB design with $v = 7$, $k = 3$ having precisely 16 distinct blocks. Clearly one could not conclude the same result if b was allowed to go beyond 21. In his Ph.D. Thesis, Foody (1979) verified this fact.

In this section we shall utilize the graph theoretical results of the previous section and demonstrate graphically that there is no BIB design based on exactly 16 distinct blocks if $v = 7$ and $k = 3$. The techniques and the ideas used here are perhaps more useful to researchers than the end result for $v = 7$ and $k = 3$.

For the rest of this section, unless specifically indicated to the contrary, all designs discussed will have $v = 7$ and $k = 3$.

4.1. Designs Containing Blocks with Frequency λ.

When $v = 7$ and $k = 3$, there is a symmetric BIB design for which $\lambda = 1$. In this section we show that all BIB designs

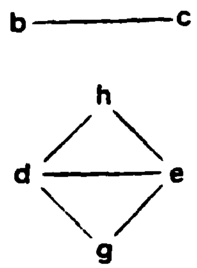
with $v = 7$ and $k = 3$ in which some block has frequency λ are unions of symmetric designs.

A special feature of the case $v = 7$, $k = 3$ which we exploit is that $b/v = \lambda$. Thus, every block with frequency λ intersects every other block in the support in the same number (clearly one) of varieties, by Corollary 3.5.

We also need the following lemma to prove our main proposition.

Lemma 4.1. Let $V = \{a,b,c,d,e,g,h\}$ and suppose D is a $BIB(7,7\lambda,3\lambda,3,\lambda)$. If $f(abc) = \lambda$ and if $\{ade\} \in D^*$, then $\{agh\} \in D^*$.

Proof: Consider the figure around variety a. By Corollary 3.16 each vertex other than b and c must have degree at least equal to 2 if $\{agh\} \not\in D^*$.



For example, h must be adjacent to either d or e, say d, but d and e already have an edge connecting them; thus $s(ad) \geq 2$ so $s(ae) \geq 2$ and $s(ah) \geq 2$. So the figure around a must be as depicted above. But this figure violates Proposition 3.15. Thus $\{agh\} \in D^*$.

Proposition 4.2. Suppose D is a BIB$(7,7\lambda,3\lambda,3,\lambda)$ and suppose $B \in D^*$ and $f(B) = \lambda$. Then D contains $D_0 =$ BIB$(7,7,3,3,1)$ and $B \in D_0^*$.

Proof: For concreteness, let $B = 123$. Note that the pairs 12, 13, and 23 can occur in no other block of D^*. Now $s(14) \geq 0$, so we assume without loss, that $145 \in D^*$. It follows by Lemma 4.1 that $167 \in D^*$. Either 246 or 247 is in D^*; for, if we suppose not, then $245 \in D^*$, since $s(24) > 0$. Then $s(45) \geq 2$, since $145 \in D^*$. So, by Lemma 3.10, $s(24) \geq 2$. Then 246 or 247 is in D^*. It can easily be seen that there is no loss in assuming that $246 \in D^*$. But Lemma 4.1 then implies that $257 \in D^*$. So we so far have in D^*.

123	167	257
145	246.	

Now if either 347 or 356 is in D^*, then so is the other, by Lemma 4.1. But these two blocks, together with the five listed above, would produce D_0 as required. So, if there is no such D_0, we can assume that neither 347 nor 356 is in D^*. But $s(34) > 0$, so 345 or 346 is in D^*, so another application of Lemma 3.10 shows that $s(34) \geq 2$, so that both 345 and 346 are in D^*. And another application of Lemma 4.1 shows that 367 and 357 must also be in D^*. So, we now have

123	167	257	346	367
145	246	345	357.	

If $156 \in D^*$ then so is 147, by Lemma 4.1 and we have constructed a D_0. Similarly if $256 \in D^*$. But $s(56) > 0$, so it must be that $456 \in D^*$. However, 456 is disjoint from 123 and this is impossible. Thus, there must be a D_0 contained in D as required.

Corollary 4.3. If D is as above, then D is the union of designs that are BIB(7,7,3,3,1).

Proof: The corollary follows immediately from the proposition by induction on λ.

Using this fact we can now give all possible support sizes for designs containing a block with frequency λ.

Proposition 4.4. If $B \in D$ such that $f(B) = \lambda$, then $b^* = 7, 11, 13, 15, 17$ or 19.

Proof: By Corollary 4.3, D^* is the union of designs D_i, $i = 1, \ldots, n$ each a BIB(7,7,3,3,1). Without loss we can assume that for no i is $D_i \subset \bigcup_{j \neq i} D_j$ and that $B = 123$. Since every block intersects B in exactly one variety, $b^* \leq 19$. Observe that $|D_i \cap D_j| = 1$ or 3, since any two blocks of a BIB(7,7,3,3,1) determine a third (Lemma 4.1)

and any four determine the whole design.

First, we show that if $|\bigcap_{i=1}^{n} D_i| = 3$ then $n = 2$ and $b^* = 11$. For, since $123 \in \cap D_i^*$, we can assume, without loss that $145 \in \cap D_i^*$. Lemma 4.1 then shows that $167 \in \cap D_i^*$. Then for any D_i, either 246 or 247 is in D_i. But four blocks determine each D_i. Thus $D^* = D_1 \cup D_2$ is the only such design.

We now show that b^* is odd, by induction on n. If $n = 1$, then $b^* = 7$. If $n = 2$, then $|D_1 \cap D_2|$ is 3 or 1 and b^* is 11 or 13. If $n > 2$, then $|\bigcap_{i=1}^{n} D_i| = 1$ by the last paragraph. Let $D^* = \bigcup_{i=1}^{n} D_i$. By the inductive hypothesis, $|\bigcup_{i=1}^{n-1} D_i|$ is odd. Consider the sets, for $i = 1,\ldots,n-1$,

$(D_n \cap D_i) - \bigcap_{j=1}^{n} D_j$. By the last paragraph, these sets are disjoint. And each of them has cardinality 0 or 2. Thus $|D_n - \bigcup_{i=1}^{n-1} D_i|$ is even, so $|D^*|$ is odd.

Corollary 4.5. *For a* BIB$(7,7\lambda,3\lambda,3,\lambda|b^*)$, *if* $b^* < 14$ *then* $b^* = 7$ *or* 11.

Proof: If $b^* < 14$ then the total number of pairs in D^* is less than 42. But there are 21 distinct pairs. Thus some pair occurs in only one block of D^*, so the frequency of this block is λ. The last proposition then says that b^* must be 7 or 11.

In Corollary 4.3, the condition that D^* contains some block with frequency λ cannot be removed. For example, in Hedayat and Li (1979), there is a uniform design with $b = b^* = 21$. Using Proposition 2.8 it can be checked that this design has a minimal support and thus, by Corollary 3.2, it is a fundamental design. It follows, then, that it cannot be the union of other designs.

4.2. There is no $BIB(7,7\lambda,3\lambda,3,\lambda|16)$.

We now direct our attention to proving that there does not exist a $BIB(7,7\lambda,3\lambda,3,\lambda|16)$. Proposition 4.2 allows us to restrict our attention to designs in which every pair of varieties occurs in at least two distinct blocks: i.e., designs in which no block has frequency λ.

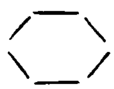
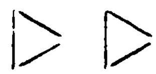
There are only five distinct blocks containing any particular pair of varieties. We proceed by considering the cases where every pair appears in exactly two or three distinct blocks, where some pair appears in four distinct blocks, and where some pair appears in five distinct blocks. We must divide the first case, where every pair appears at most in three distinct blocks, into subcases which depend on in how many distinct blocks a single variety may occur. We begin by determining possible values of $s(a)$, where a is a variety.

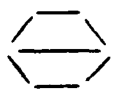
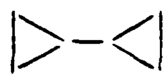
Lemma 4.6. Suppose for every pair, X, of varieties in D, $s(X) \geq 2$. If $s(ab)$ is odd, then there exists a variety $c \neq b$ such that $s(ac)$ is odd. Further, $s(a) \geq 7$.

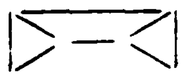
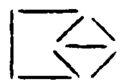
Proof: Consider the figure around variety a. The vertex b has odd degree. If the other five vertices all have even degree, then the total figure has odd degree, which is impossible. Thus, one other vertex must have degree of at least 3, giving at least a total degree of 14 for the figure; that is, at least 7 blocks.

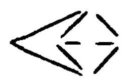
We now exhibit all possible figures around a variety if every pair occurs in two or three distinct blocks.

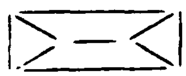
Proposition 4.7. Suppose $2 \leq s(ab) \leq 3$ for every pair of varieties in D. Then $6 \leq s(a) \leq 9$ and the figure around a is:

A) If $s(a) = 6$: (i)  or (ii) 

B) If $s(a) = 7$: (i)  or (ii) 

C) If $s(a) = 8$: (i)  or (ii) 

or (iii) 

D) If $s(a) = 9$: (i)  or (ii) 

<u>Proof</u>: Variety a appears in six distinct pairs, each of which occurs in at least two blocks of D^*. Exactly two of these pairs fit in any one block, so $s(a) \geq 6$. Similarly, if $s(ab) \leq 3$ for all b then the, at most, 18 pairs containing a occur in at most nine blocks. For specificity, let $a = 1$ for the rest of the proof.

(i) If $s(1) = 6$, then every pair of varieties containing the variety 1 must occur exactly twice in D^*; that is, every vertex in the figure around a must have exactly two adjacent vertices. Without loss we can start the figure around 1 as

$$2 \text{ --- } 3 \text{ --- } 4.$$

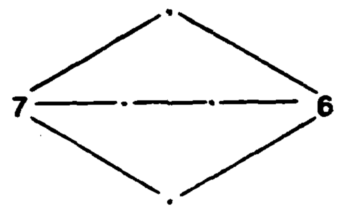
Now if $124 \in D^*$, then the figure can only be completed as in A(ii). On the other hand, if $124 \notin D^*$, then we can take 4 as adjacent to 5.

$$2 \text{ --- } 3 \text{ --- } 4 \text{ --- } 5.$$

If 5 is adjacent to 2, then 6 and 7 will only have degree one. So, assume $156 \in D^*$. The only way to give to 7 two adjacencies while using only six edges is to include 167 and 126 in D^*, thereby constructing figure A(i).

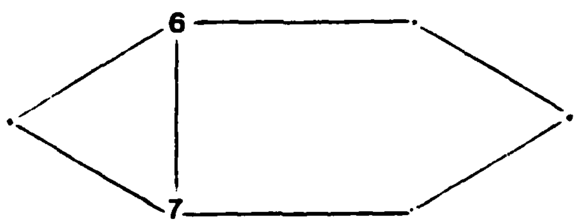
(ii) If $s(1) = 7$, then two vertices in the figure around 1 have degree three and the other four have degree two. We can assume that the two vertices with degree three correspond to varieties 6 and 7.

If 6 and 7 are not adjacent, then they are both adjacent to the same two vertices. Therefore, the only possible figure around 1 is of the form:



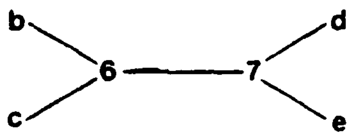
But this figure violates Proposition 3.15.

On the other hand, suppose $167 \in D^*$. If there is a vertex adjacent to both 6 and 7, then the only possible figure around 1 is:



But this figure also violates Proposition 3.15.

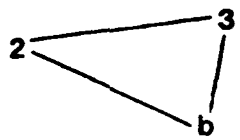
So we are left with:



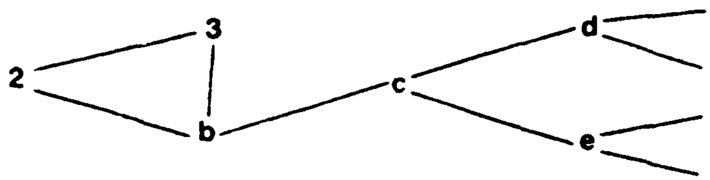
If we make b and c adjacent, we get B(ii). On the other hand, if we make b and d or b and e adjacent, we get B(i).

(iii) If $s(1) = 6$, then in the figure around 1, two of the vertices have degree two and the rest have degree three. Let 2 and 3 be the vertices of degree two.

Suppose first that 2 and 3 are adjacent to each other. *If they are also both adjacent to the same vertex, say*

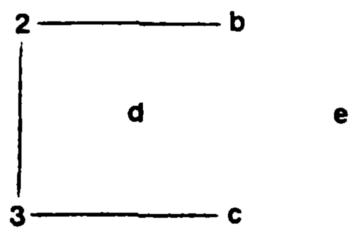


then we can only continue this figure as

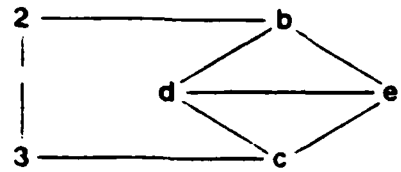


But *we cannot complete this figure, while still giving both* d and e degree three.

Thus, 2 and 3 are not adjacent to the same vertex. So we have:



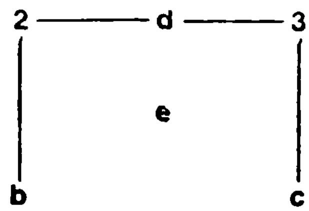
Now e must be adjacent to b, c, and d; and d must be adjacent to c, b, and e, giving us figure C(ii):



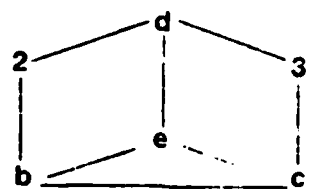
Suppose on the other hand, that 2 and 3 are not adjacent to each other, and that the distance between them is one.

$$2 \text{ — } d \text{ — } 3.$$

If there is another vertex adjacent to both 2 and 3, then we have violated Proposition 3.16. Thus we have:



Now, b, c, d, and e all have degree three, so we must end up with:

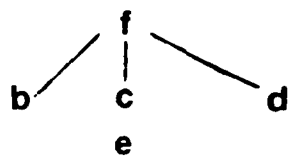


But this figure violated Proposition 3.15, as was shown in the example following that proposition.

Suppose now that 2 and 3 are a distance of two apart.

If e is adjacent to c, we get figure C(i). If e is adjacent to d, we get figure C(iii).

(iv) Suppose $s(1) = 9$. Every vertex of the figure around 1 must have degree three. We begin the figure with



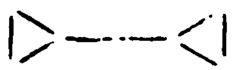
Now, e must be adjacent to at least two of b, c, or d, since e has degree three and there are only six vertices. If e is adjacent to two of the three, figure D(i) results. If e is adjacent to three of them, then figure D(ii) results.

We now begin ruling out the cases in which no pair occurs in more than three blocks of D^*. We start with the subcases in which no variety occurs in more than seven blocks of D^*.

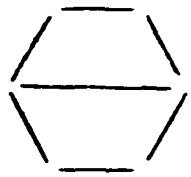
<u>Proposition 4.8</u>. <u>Suppose</u> $2 \leq s(ab) \leq 3$ <u>for every pair of varieties in</u> D, <u>and suppose</u> $s(a) \leq 7$ <u>for every variety in</u> D: <u>then</u> $b^* \neq 16$.

Proof: Suppose that $b^* = 16$. Then $\sum_a s(a) = 48$. But $2 \leq s(ab)$ for all pairs ab, so for each variety, $s(a) \geq 6$. Thus, $s(a) = 7$ for all varieties except one, say variety 7, and $s(7) = 7$.

Therefore, if $a \neq 7$, the figure around a must be either

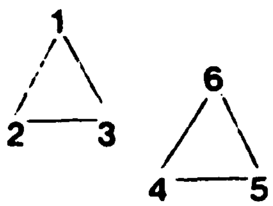


or

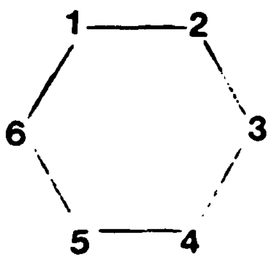


by Proposition 4.7. By the same proposition, the figure around 7 must be either

Case A:



or Case B:



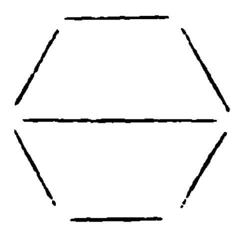
Consider the 15 pairs from varieties 1,...,6. We claim that those of these 15 pairs defined by the figure around 7 (e.g., 12, 23,...) occur in exactly three blocks of D^*.

First note that $\sum_{a<b\le 6} s(ab) = 36$ and $2 \le s(ab) \le 3$ for all pairs, so that six pairs do occur in three blocks and the other nine in two blocks. To prove the claim it suffices to show that no pair defined by the figure around 7 occurs in only two blocks of D^*.

Suppose the opposite; that is, suppose without loss of generality, that $s(13) = 2$.

First let us suppose that the figure around 7 is as in <u>Case A</u> above. In the figure around 1, 7 has degree two and is adjacent to both 2 and 3. But the figure around 1 is either

or

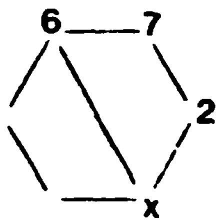


by Proposition 4.7. But $s(13) = 2$, so $s(12) = 3$.

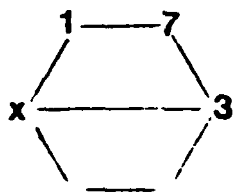
By the same arguement applied to the figure around 3, it follows that $s(23) = 3$. So when we draw the figure around 2, we find that both 1 and 3 have degree three and that 7 must be adjacent to both 1 and 3. But this is not possible in either of the permitted figures around 2, since $s(2) = 7$. We have thus shown a contradiction if the figure around 7 is as in Case A.

Now suppose the figure around 7 is as in Case 3. We will consider three subcases.

(1) Suppose that the figures for 1 and 2 are both 
In the figure around 1, either 2 or 6 must have degree three, since 7 does not and since 2 and 6 are adjacent to 7. So without loss the figure around 1 is:



Then both 1 and 7 have degree two in the figure around 2, so in this figure, x must be adjacent to 3:



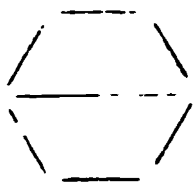
Thus x must be 4 or 5.

If x = 4, then the figure around 4 contains:

Similarly, if x = 5, then the figure around 5 contains:

But these figures cannot be completed as legal figures with seven edges.

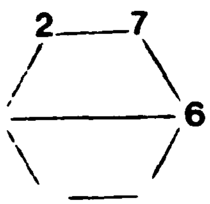
(2) Suppose the figures around 1 and 2 are respectively:



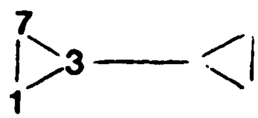
and

.

Then, again supposing without loss of generality that 6 rather than 2 has degree three, we have as the figures around 1 and 2 respectively:



and

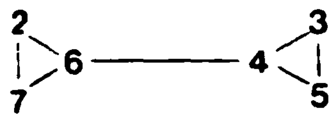


.

Thus 3 must be adjacent to 2 in the figure around 1; but this implies that $s(13) = 3$, contradicting the premise.

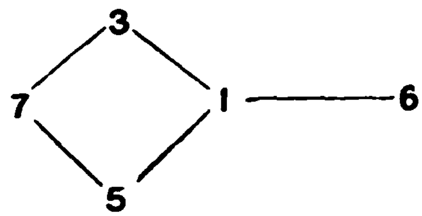
(3) Suppose that the figure around 1 is:

Again, we can assume that $s(16) = 3$, so the figure around 1 must be

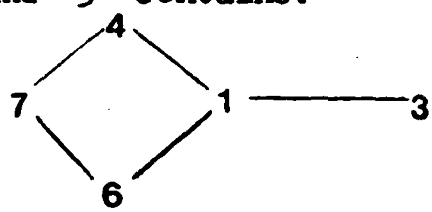


or

In the first case, the figure around 4 contains:

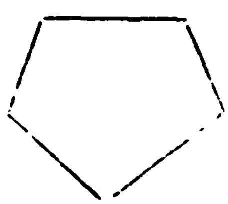


which cannot be completed as a legal figure. In the second case, the figure around 5 contains:

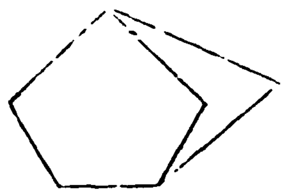


which is subject to the same objection. Thus, the claim is proven.

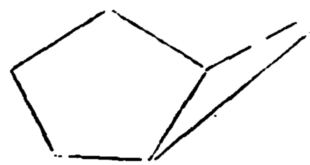
This claim implies that the remaining ten blocks of D^* that do not contain variety 7 form a BIB(6,10,5,3,2). It is known that this design is unique up to isomorphism. Denote these ten blocks of D^* by S. Now the figure around any variety in S is:



For definiteness, consider the figure around 1 in S. Now 1 is adjacent to two varieties in the figure around 7, so the figure around 1 in D^* is either:



or



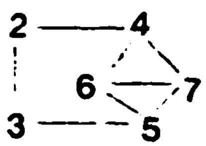
.

But both of these violate Proposition 3.16. Thus, there can be no BIB($7,7\lambda,3\lambda,3,\lambda|16$) of the type described in this proposition.

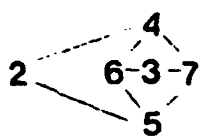
<u>Proposition 4.9.</u> <u>Suppose</u> $2 \leq s(ab) \leq 3$ <u>for all pairs of varieties in</u> D, <u>and suppose</u> $s(1) = 8$; <u>then</u> $b^* \neq 16$.

<u>Proof</u>: The three possible figures around variety 1 are, without loss of generality:

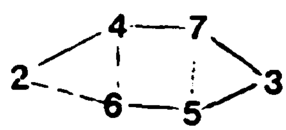
A)



or B)



or C)



In any of these case, $s(14) = s(15) = s(16) = s(17) = 3$. But by Lemma 4.6, each of the varieties 4, 5, 6, and 7 must occur in another pair which appears in three distinct blocks of D^*. But by counting to $\Sigma\, s(ab) = 48$, there can be at most six pairs repeated in three blocks of D^*. Thus

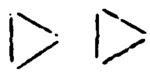
the pairs repeated three times are, in addition to the four listed above:

(i) 45 and 67, or

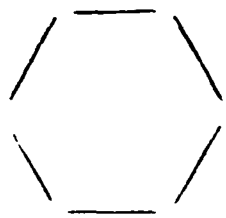
(ii) 46 and 57, or

(iii) 47 and 56.

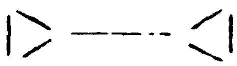
Note also that varieties 2 and 3 occur only in pairs occurring in two blocks of D^*. Thus, the figures 2 and 3 have six edges and are thus either



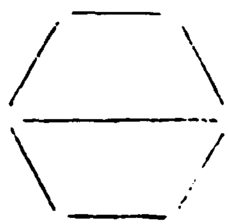
or



and the figures around 4, 5, 6, and 7 have seven edges each, and are thus



or



Note that in these figures around 4, 5, 6, and 7, there are two vertices of degree three, that no vertex is adjacent to both of the vertices of degree three, and that every vertex is adjacent to at least one vertex of degree three.

Suppose now that (i) holds; i.e., that

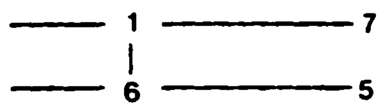
$$s(45) = s(67) = 3.$$

Then vertices 1 and 5 have degree three in the figure around 4. But 4 and 5 are not adjacent in the figure around 1, so 1 and 5 are not adjacent in the figure around 4, even though they both have degree three. This is impossible.

Suppose instead that (ii) holds; i.e., that

$$s(46) = s(57) = 3.$$

Then in the figure around 4, 1 and 6 have degree one, and 5 is not adjacent to 1, since 4 and 5 are not adjacent in the figure around 1. Thus, in the figure around 4, vertex 5 must be adjacent to the other vertex of degree three, namely 6. That is, the figure around 4 contains:



Now we can examine the figures around 1 and 4 to see that, in the figure around 6, both 1 and 4 are adjacent to 5. But 1 and 4 have degree three in the figure around 6, producing a contradiction.

Thus, (ii) cannot hold, and inspection of the figures around variety 1 shows that the remaining case (iii) is symmetrical to (ii) and that the same argument would apply. Therefore no BIB with the properties listed in the statement of the proposition can exist.

There now remains the subcase that $s(a) = 9$ for some variety. The next proposition will help us eliminate both this subcase and the case that some pair occurs in five blocks of D^*.

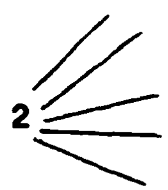
Proposition 4.10. If $s(ab) \geq 2$ *for all pairs* ab *in* D, *and if* $s(12)$ *is odd and* $s(1) \geq 9$, *then* $b^* > 16$.

Proof: If $b^* \leq 16$ then $\sum_{i<j} s(ij) \leq 48$. But if $s(1) > 9$ then $\sum_{i} s(1i) > 18$ and $\sum_{i<j} s(ij) > 48$, since there are 15 pairs not containing variety 1 and each occurs in at least two blocks of D^*. So, $s(1) = 9$ and $\Sigma(1i) = 18$. But by Lemma 4.6 the fact that $s(12)$ is odd shows that there exists a variety $c \neq 1$ such that $s(c2) > 2$. So $\sum_{1<i<j} s(ij) > 30$ and $\sum_{i<j} s(ij) > 48$, a contradiction.

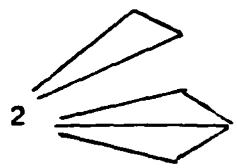
Corollary 4.11. If $2 \leq s(ab) \leq 3$ *for all pairs* ab *in* D, *and if* $s(1) = 9$, *then* $b^* > 16$.

Corollary 4.12. If $2 \leq s(ab)$ *for all pairs* ab *in* D, *and if* $s(12) = 5$, *then* $b^* > 16$.

Proof: If $s(12) = 5$, then the figure around 1 contains:



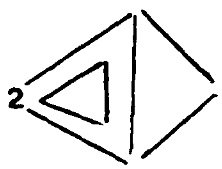
Now every vertex has degree greater than or equal to two, so the only way to complete the figure in eight or less lines is:



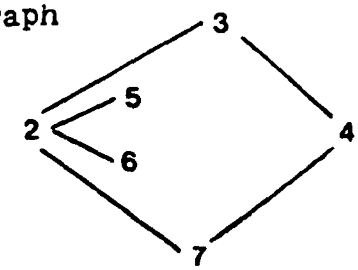
But this violates Proposition 3.16. Thus, $s(1) \geq 9$, and so by the last proposition, $b^* > 16$.

There now remains only the case in which some pair occurs in four blocks of D^*. The next two results settle this case.

Proposition 4.13. Suppose $s(ab) \geq 2$ for all pairs ab in D and suppose $s(12) = 4$ and $s(1) \leq 8$. Then the figure around variety 1 is:



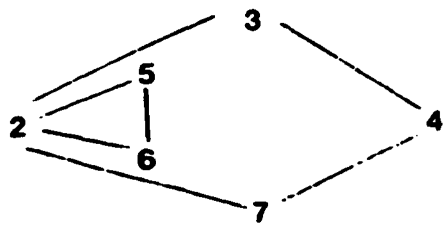
Proof: Without loss, we can assume the figure around 1 contains the subgraph



The figure cannot be completed with just one more line, since

that line would have to join 5 and 6, and the resulting figure would violate Proposition 3.16.

One of the two lines to be added must still join 5 and 6, since any pair of lines, one containing 5 and the other 6, will form a figure violating Proposition 3.16. So, we now must add one more line to the figure below:



Adjoining 5 or 6 to 4 violates Proposition 3.16. Adjoining 5 to 3 violates Proposition 3.15 (consider the subgraph formed by removing line 2-5). Thus, the only possible figure is the one claimed.

<u>Proposition 4.14. If</u> $2 \leq s(ab)$ <u>for all pairs</u> ab <u>in</u> D, <u>and if</u> $s(12) = 4$, <u>then</u> $b^* > 16$.

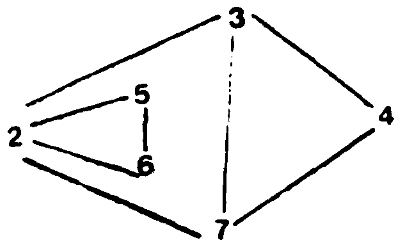
<u>Proof:</u> If $s(1) + s(2) \geq 17$, then

$$s(12) + \sum_{j>2} (s(1j) + s(2j)) \geq 34-4 = 30.$$ But then

$$\sum_{a<b} s(ab) \geq 30 + \sum_{2<a<b} s(ab) \geq 50$$ which cannot be if $b^* \leq 16$.

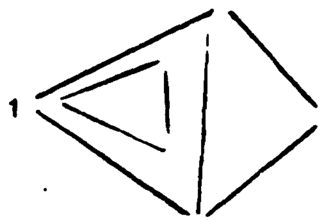
So $s(1) \leq 8$ or $s(2) \leq 8$ and by Proposition 4.13, $s(1) = s(2) = 8$ and the figures around varieties 1 and 2 are without loss

(1)

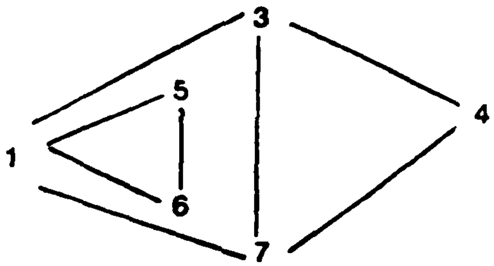


and

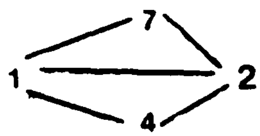
(2)



Now if 5 (or 6) has degree 3 in the figure around 2, then Lemma 4.6 implies that there exists variety $c \notin \{1,2\}$ such that $s(3c) \geq 3$. This then gives $\sum_{a<b} s(ab) > 48$. So, the figure around 2 must be



Then the figure around 3 contains



and must then contain a vertex of degree 3 other than variety 1 or 2. So again $\sum_{a<b} s(ab) > 48$.

Since all cases have been eliminated, we have proven:

<u>Proposition 4.15.</u> <u>There does not exist a</u> $BIB(7,7\lambda,3\lambda,3,\lambda|16)$.

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The structure and the size of the supports of balanced incomplete block (BIB) designs are explored. The concept of fundamental BIB designs is introduced and its usefulness in the study of the support of BIB designs is demonstrated. It is shown that the support size can be reduced via a technique called trade on a design. A new graphical method of studying the supports of BIB designs with blocks of size three is introduced. Several useful results are obtained via this graphical method. In particular, it is shown that no BIB design with seven varieties in blocks of size three can be built based on sixteen distinct blocks. Contributions made here have immeidate applications in controlled experimental designs and survey samplings.

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